ISRAEL JOURNAL OF MATHEMATICS, Vol. 40, No. 2, 1981

RANDOM EMBEDDINGS OF EUCLIDEAN SPACES IN SEQUENCE SPACES

BY

GIDEON SCHECHTMAN[†]

ABSTRACT

It is proved that for some absolute constant d and for $n \leqq dm$ most $n \times m$ matrices with ± 1 entries are good embeddings of l_2^n into l_1^m. Similar theorems are obtained where l_1^m is replaced by members of a wide class of sequence spaces.

1. Introduction

The main result of this paper is the following theorem.

THEOREM 1. *There exists a constant $d > 0$ such that for all pairs of natural numbers n, m with $n \leqq dm$ there exist signs $\varepsilon_{i,j} = \pm 1$, $i = 1, \cdots, m$, $j = 1, \cdots, n$ satisfying*

$$\frac{1}{4}\left(\sum_{j=1}^{n} a_j^2\right)^{1/2} \leqq \left\| \sum_{j=1}^{n} a_j f_j \right\| \leqq 4\left(\sum_{j=1}^{n} a_j^2\right)^{1/2} \quad \textit{for all } (a_j)_{j=1}^{n} \subseteq \mathbf{R}$$

where $f_j = (1/m)(\varepsilon_{1,j}, \varepsilon_{2,j}, \cdots, \varepsilon_{m,j})$ and $\|\cdot\|$ is the l_1^m norm. Moreover, the statement holds for more than half the possible choices of signs $\varepsilon_{i,j}$.

Of course, the novelty of this theorem is in the particular form of embedding l_2^n into l_1^m. The fact that l_2^n embeds into l_1^m for $n \leqq dm$ is by now well-known (cf. [3] and [6]). We also find the proof of Theorem 1 instructive since it reveals the relation between the central limit theorem and finding euclidean subspaces in Banach spaces.

A similar theorem is proved in [2] and [1] for l_s^m, $2 \leqq s < \infty$ instead of l_1^m; a combination of the two results enables us to extend them to a large class of spaces. We first recall the definition of q-concavity: Let X be a space with a 1-unconditional basis $x_1, \cdots, x_m$, X is said to have *q-concavity constant* $\leqq M$ ($1 \leqq q \leqq \infty$, $1 \leqq M < \infty$) if for all n and for all f_j in X, $1 \leqq j \leqq n$

[†] Supported in part by NSF Grant No. MCS-79-03042.
Received May 3, 1981

$$\left\|\left(\sum_{j=1}^{n}|f_j|^q\right)^{1/q}\right\| \geqq M^{-1}\left(\sum_{i=1}^{n}\|f_i\|^q\right)^{1/q}$$

where

$$|f|^\alpha = \sum_{i=1}^{m}|a_i|^\alpha x_i \quad \text{if } f=\sum_{i=1}^{m}a_i x_i.$$

THEOREM 2. *For all $1 \leqq q, M < \infty$ there exists a $d = d(q,M) > 0$ such that if n and m satisfy $n \leqq d \cdot \min(m, m^{2/q})$ and $x_1, \cdots, x_m$ is a 1-symmetric basis of some Banach space with q-concavity constant $\leqq M$ then there exist signs $\varepsilon_{i,j} = \pm 1$, $i=1,\cdots,m$, $j=1,\cdots,n$ satisfying*

$$d\left(\sum_{j=1}^{n}a_j^2\right)^{1/2} \leqq \left\|\sum_{j=1}^{n}a_j f_j\right\| \leqq d^{-1}\left(\sum_{j=1}^{n}a_j^2\right)^{1/2} \quad \textit{for all } (a_j)_{j=1}^n \subseteq \mathbf{R}$$

where $f_j = \lambda(m)^{-1}\sum_{i=1}^m \varepsilon_{i,j}x_i$ $\quad(\lambda(m) = \|\sum_{i=1}^m x_i\|)$.

THEOREM 3. *For all $1 \leqq q, M < \infty$ there exists a $d = d(q,M) > 0$ such that if n and m satisfy $n \leqq d \cdot \min(m, m^{2/q})$ and $x_1, \cdots, x_m$ is a 1-unconditional basis of some Banach space with q-concavity constant $\leqq M$ then there exist signs $\varepsilon_{i,j} = \pm 1$, $i=1,\cdots,m$, $j=1,\cdots,n$ and a sequence $\alpha_1,\cdots,\alpha_m$ in $\mathbf{R}$ such that*

$$d\left(\sum_{j=1}^{n}a_j^2\right)^{1/2} \leqq \left\|\sum_{j=1}^{n}a_j f_j\right\| \leqq d^{-1}\left(\sum_{j=1}^{n}a_j^2\right)^{1/2} \quad \textit{for all } (a_j)_{j=1}^n \subseteq \mathbf{R}$$

where $f_j = \sum_{i=1}^m \alpha_i \varepsilon_{i,j} x_i$.

Theorem 1 is proved in Section 2, and Theorems 2 and 3 in Section 3.

2. Proof of Theorem 1

Let $(\Omega, \mathcal{F}, P)$ be a probability space, $(\phi,\Omega) = \mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots \mathcal{F}_l = \mathcal{F}$ an increasing sequence of σ-fields. Given an $\mathcal{F}$ measurable real function f we denote by $E_k f$ the conditional expectation of f with respect to $\mathcal{F}_k$, $0 \leqq k \leqq l$, and by $d_k = E_k f - E_{k-1} f$, $1 \leqq k \leqq l$ the martingale difference sequence associated with f.

LEMMA 1. *Assume $E_{k-1}d_k^2 \leqq \alpha_k^2$ and $|d_k| \leqq \beta$ for $1 \leqq k \leqq l$. Then for all $0 \leqq c \leqq \beta^{-1}\sum_{k=1}^l \alpha_k^2$*

$$P(|f - E_0 f| \geqq c) \leqq 2\exp\left(-c^2\Big/\left(4\sum_{k=1}^{l}\alpha_k^2\right)\right).$$

The lemma and its proof below are probably known. However, since we could not find an adequate reference we include a proof.

PROOF. For all $|y| \leqq \beta$, $\lambda > 0$ we have the elementary inequality

$$\exp(\lambda y) \leqq 1 + \lambda y + \beta^{-2}(\exp(\lambda\beta) - 1 - \lambda\beta)y^2.$$

If in addition $\lambda\beta \leqq \frac{1}{2}$, then $\exp(\lambda\beta) - 1 - \lambda\beta \leqq (\lambda\beta)^2$ and

$$\exp(\lambda y) \leqq 1 + \lambda y + \lambda^2 y^2.$$

It follows that, for $1 \leqq k \leqq l$ and λ such that $\lambda\beta \leqq \frac{1}{2}$,

$$E_{k-1}\exp(\lambda d_k) \leqq 1 + \lambda^2\alpha_k^2 \leqq \exp(\lambda^2\alpha_k^2).$$

For each $m \leqq l$ we get

$$E\exp\left(\lambda\sum_{k=1}^{m} d_k\right) = E\prod_{k=1}^{m}\exp(\lambda d_k) = E\left[\left(\prod_{k=1}^{m-1}\exp(\lambda d_k)\right)(E_{m-1}\exp(\lambda d_m))\right]$$
$$\leqq E\left(\prod_{k=1}^{m-1}\exp(\lambda d_k)\right)\cdot\exp\lambda^2\alpha_m^2.$$

A simple induction argument shows then that, if $\lambda\beta \leqq \frac{1}{2}$,

$$E\exp\left(\lambda\sum_{k=1}^{l} d_k\right) \leqq \exp\left(\lambda^2\sum_{k=1}^{l}\alpha_k^2\right),$$

$$P\left(\sum_{k=1}^{l} d_k \geqq c\right) = P\left(\exp\left(\lambda\sum_{k=1}^{l} d_k - \lambda c\right) \geqq 1\right)$$
$$\leqq E\exp\left(\lambda\sum_{k=1}^{l} d_k - \lambda c\right) \leqq \exp\left(\lambda^2\sum_{k=1}^{l}\alpha_k^2 - \lambda c\right).$$

Choose $\lambda = c/(2\sum_{k=1}^{l}\alpha_k^2)$. Then, if $c \leqq \beta^{-1}\sum_{k=1}^{l}\alpha_k^2$, $\lambda\beta \leqq \frac{1}{2}$, and

$$P(f - E_0 f \geqq c) \leqq \exp\left(-c^2\Big/\left(4\sum_{k=1}^{l}\alpha_k^2\right)\right).$$

Since the same inequality holds for $-f$ instead of f we get the desired inequality. □

REMARK. A similar inequality has already been used in the context of Banach spaces by Maurey [8].

PROOF OF THEOREM 1. Fix $n, m \in \mathbf{N}$. Let $\phi = \sigma_0 \subset \sigma_1 \subset \cdots \subset \sigma_{n,m}$ be an increasing family of subsets of $\{1, 2, \cdots, m\} \times \{1, 2, \cdots, n\}$ with $\bar{\bar{\sigma}}_k = k$, $0 \leqq k \leqq nm$. Consider the probability space $(\Omega, \mathcal{F}, P)$ where $\Omega = \{-1, 1\}^{nm}$, $\mathcal{F} = 2^{\Omega}$ and $P((\varepsilon_{i,j})_{i=1,j=1}^{m\ \ n}) = 2^{-nm}$. For $0 \leqq k \leqq nm$ let $\mathcal{F}_k$ be the field consisting of all sets which depend only on the coordinates appearing in σ_k, i.e. an atom of $\mathcal{F}_k$ is a set of the form

$$A((\delta_{i,j})_{i,j\in\sigma_k})=\{(\varepsilon_{i,j})_{i=1,j=1}^{m\ \ n};\ \varepsilon_{i,j}=\delta_{i,j}\text{ for } i,j\in\sigma_k\}.$$

Fix a sequence $(a_j)_{j=1}^n$ with $\Sigma_{j=1}^n a_j^2=1$ and consider the function

$$f(\varepsilon)=\left\|\frac{1}{m}\sum_{i=1}^m\sum_{j=1}^n \varepsilon_{i,j}a_je_i\right\|.$$

If $\varepsilon,\delta\in\Omega$ are different only in their i,j coordinate, then $|f(\varepsilon)-f(\delta)|\leqq(2/m)|a_j|$. It follows that, if A is an atom of $\mathcal{F}_{k-1}$, then

$$\max_A E_kf-\min_A E_kf\leqq(2/m)|a_j|$$

where j is such that $\{(i,j)\}=\sigma_k\setminus\sigma_{k-1}$ for some (unique) i. Thus $|d_k|\leqq(2/m)|a_j|\leqq 2/m$. $\Sigma_{i=1}^m\Sigma_{j=1}^n((2/m)a_j)^2=4/m$, so by Lemma 1

$$P(|f-Ef|\geqq c)\leqq 2\exp(-c^2m/16)$$

for all $c\leqq 2$. By the Khinchine inequality [10], $1/\sqrt{2}\leqq Ef\leqq 1$; thus

$$P(1/2\sqrt{2}\leqq f\leqq 2)>1-2\exp(-m/128).$$

We now choose an ε-net M on the sphere of l_2^n of cardinality $\leqq\exp(2n/\varepsilon)$ (cf. [3]), and get

$$P\left(\frac{1}{2\sqrt{2}}\leqq\left\|\frac{1}{m}\sum_{i=1}^m\sum_{j=1}^n\varepsilon_{i,j}a_je_i\right\|\leqq 2\text{ for all }(a_j)\in M\right)>1-2\exp\left(\frac{2n}{\varepsilon}-\frac{m}{128}\right).$$

Now, choose ε such that

$$\frac{1}{2\sqrt{2}}\leqq\left\|\frac{1}{m}\sum_{i=1}^m\sum_{j=1}^n\varepsilon_{i,j}a_je_i\right\|\leqq 2\qquad\text{for all }(a_j)_{j=1}^n\in M$$

implies

$$\frac{1}{4}\leqq\left\|\frac{1}{m}\sum_{i=1}^m\sum_{j=1}^n\varepsilon_{i,j}a_je_i\right\|\leqq 4\quad\text{for all }(a_j)_{j=1}^n\text{ with }\sum_{j=1}^n a_j^2=1.$$

Then, if $n\leqq\frac{1}{2}\varepsilon(m/128-\log 4)$,

$$\frac{1}{4}\leqq\left\|\frac{1}{m}\sum_{i=1}^m\sum_{j=1}^n\varepsilon_{i,j}a_je_i\right\|\leqq 4$$

$$\text{for all }(a_j)_{j=1}^n\text{ with }\sum_{j=1}^n a_j^2=1\text{ with probability }>\frac{1}{2}.$$ □

Remark. The requirement that $(e_i)_{i=1}^m$ is the unit vector basis of l_1^m and that $\|\cdot\|$ is the l_1^m norm can be weakened somewhat in one of the following two ways: (1) $\exists K$ such that

$$\operatorname*{Ave}_{\pm}\left\| \sum_{i=1}^{m} \pm a_i e_i \right\| \overset{K}{\approx} \sum_{i=1}^{m} |a_i| \qquad \text{for all } (a_i)_{i=1}^{m},$$

or (2) $\|\ \ \|$ has cotype $q < \infty$ with constant L, $\|e_i\| = 1$ for $1 \leqq i \leqq m$ and

$$K^{-1} \leqq \operatorname*{Ave}_{\pm}\left\| \sum_{i=1}^{m} \pm e_i \right\| \Big/ m \leqq K.$$

The constants d and 4 in the statement of the theorem will depend now on K, L and q.

3. Proof of Theorems 2 and 3

We begin by stating a theorem proved in [2] and [1]:

THEOREM (B.D.G.J.N.). *For each $2 \leqq q < \infty$ there exist constants $0 < K$, $d < \infty$ such that for all n and m with $n \leqq dm^{2/q}$ for more than half the possible choices of signs $\varepsilon_{i,j} = \pm 1$, $i = 1, \cdots, m$, $j = 1, \cdots, n$*

$$K^{-1}\left(\sum_{j=1}^{n} a_j^2\right)^{1/2} \leqq \left\| \sum_{j=1}^{n} a_j f_j \right\| \leqq K\left(\sum_{j=1}^{n} a_j^2\right)^{1/2} \qquad \textit{for all } (a_j)_{j=1}^{n} \subseteq \mathbf{R}$$

where $f_j = m^{-1/q}(\varepsilon_{1,j}, \varepsilon_{2,j}, \cdots, \varepsilon_{m,j})$ and $\|\cdot\|$ is the l_q^m norm.

This is essentially theorem 1.1 in [1]. The case $q = 2$ is not included in the statement there but see the remark following lemma 2.3 in [1]. For $q > 2$, d can be chosen as 1.

As an immediate corollary of this and Theorem 1 we get:

PROPOSITION 1. *Let $2 \leqq q < \infty$. There exist constants $0 < d, K < \infty$ such that if $n \leqq dm^{2/q}$ and $x_1, \cdots, x_m$ is a sequence in some Banach space satisfying*

$$(*) \qquad \frac{1}{m}\sum_{i=1}^{m} |a_i| \leqq \left\| \sum_{i=1}^{m} a_i x_i \right\| \leqq \left(\frac{1}{m}\sum_{i=1}^{m} |a_i|^q\right)^{1/q} \qquad \textit{for all } (a_i)_{i=1}^{m} \subseteq \mathbf{R}$$

then there exist signs $\varepsilon_{i,j}$, $i = 1, \cdots, m$, $j = 1, \cdots, n$ with

$$K^{-1}\left(\sum_{j=1}^{n} a_j^2\right)^{1/2} \leqq \left\| \sum_{j=1}^{n} a_j f_j \right\| \leqq K\left(\sum_{j=1}^{n} a_j^2\right)^{1/2}$$

where $f_j = \sum_{i=1}^{m} \varepsilon_{i,j} x_i$, $j = 1, \cdots, n$.

Moreover, there is one sequence of signs $(\varepsilon_{i,j})$ which works simultaneously for all $(x_i)_{i=1}^{m}$ satisfying $(*)$.

PROOF OF THEOREM 2. If $(x_i)_{i=1}^{m}$ is 1-symmetric, then

$$(**)\qquad \frac{1}{m}\sum_{i=1}^{m}|a_i| \leqq \lambda(m)^{-1}\left\|\sum_{i=1}^{m} a_i x_i\right\| \leqq \max_{1\leqq i\leqq m}|a_i|.$$

A simple renorming argument (cf. [7], p. 54) shows that without loss of generality the q-concavity constant $M = 1$. Then the argument in ([5], p. 14) shows that the right-hand side in (**) may be replaced by $((1/m)\Sigma_{i=1}^{m}|a_i|^q)^{1/q}$ and we apply the proposition. □

REMARK. The assumption "$(x_i)_{i=1}^{m}$ is 1-symmetric" can be replaced by "$(x_i)_{i=1}^{m}$ is 1-unconditional and $\|\Sigma_{i=1}^{m} x_i\|\,\|\Sigma_{i=1}^{m} x_i^*\| = m$"; consult [9] where such bases are treated.

PROOF OF THEOREM 3. It follows from [4] that there exists a sequence $(\alpha_i)_{i=1}^{m}$ such that

$$\frac{1}{m}\sum_{i=1}^{m}|a_i| \leqq \left\|\sum_{i=1}^{m} a_i\alpha_i x_i\right\| \leqq \max_{1\leqq i\leqq m}|a_i| \qquad \text{for all } (a_i)_{i=1}^{m}\subseteq \mathbf{R}.$$

The rest of the proof is exactly the same as that of Theorem 2. □

REFERENCES

1. G. Bennett, L. E. Dor, V. Goodman, W. B. Johnson and C. M. Newman, *On uncomplemented subspaces of L_p, $1 < p < 2$*, Israel J. Math. **26** (1977), 178–187.

2. G. Bennett, V. Goodman and C. M. Newman, *Norms of random matrices*, Pacific J. Math. **59** (1975), 359–365.

3. T. Figiel, J. Lindenstrauss and V. D. Milman, *The dimension of almost spherical sections of convex bodies*, Acta Math. **139** (1977), 53–94.

4. R. I. Jamison and W. H. Ruckle, *Factoring absolutely converging series*, Math. Ann. **224** (1976), 143–148.

5. W. B. Johnson, B. Maurey, G. Schechtman and L. Tzafriri, *Symmetric structures in Banach spaces*, Memoirs Amer. Math. Soc. No. **217**, 1979.

6. B. S. Kashin, *Diameter of some finite dimensional sets and of some classes of smooth functions*, Izv. AN SSSR, Ser. Mat., **41** (1977), 334–351 (Russian).

7. J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces, II, Function Spaces*, Springer-Verlag, 1979.

8. B. Maurey, *Construction de suites symmetrique*, C. R. Acad. Sci. Paris, Sér. A-B **288** (1979), 679–681.

9. H. P. Rosenthal, *Some applications of p-summing operators to Banach space theory*, Studia Math. **58** (1976), 21–43.

10. S. I. Szarek, *On the best constant in the Khinchine inequality*, Studia Math. **58** (1976), 197–208.

DEPARTMENT OF THEORETICAL MATHEMATICS
THE WEIZMANN INSTITUTE OF SCIENCE
REHOVOT 76100, ISRAEL